# Feedback Control of Dynamic Systems

Fourth Edition

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As long as the commanded values of $x$ produce $\theta$ motion that has a sufficiently small value of $\ddot{\theta}$, the approximation given by Eqs. (2.99) or (2.100) is valid and no other linearized dynamic relationships are necessary. However, as soon as the commanded values of $x$ produce accelerations where the inertial forces ($m\ddot{y}$ and the reaction to $I\ddot{\theta}$) are a significant fraction of $p_s - p_e$, the approximations are no longer valid. We must then incorporate these forces into the equations, thus obtaining a dynamic relationship between $x$ and $\theta$ that is much more involved than the pure integration implied by Eqs. (2.99) or (2.100). Typically, for initial control system designs, hydraulic actuators are assumed to obey the simple relationship of Eqs. (2.99) or (2.100).

## 2.6 Linearization and Scaling

The differential equations of motion for almost all processes selected for control are nonlinear. On the other hand, as will be evident in the next chapter, both analysis and control design are far easier for linear than for nonlinear models. **Linearization** is the process of finding a linear model that approximates a nonlinear one. Fortunately, as Lyapunov proved over 100 years ago, if a small-signal linear model is valid near an equilibrium and is stable, then there is a region (which may be small, of course) containing the equilibrium within which the nonlinear system is stable.[11] So we can safely make a linear model and design a linear control for it such that, at least in the neighborhood of the equilibrium, our design will be stable. Because a very important role of feedback control is to maintain the process variables near equilibrium, such small-signal linear models are a frequent starting point for control models. Small-signal linearization is discussed in Section 2.6.1.

An alternative approach to obtain a linear model for use as the basis of control system design is to use part of the control effort to cancel the nonlinear terms and to design the remainder of the control based on linear theory. This approach—linearization by feedback—is popular in the field of robotics, where it is called the **method of computed torque**. It is also a research topic for control of aircraft. Section 2.6.2 takes a brief look at this method. Finally, some nonlinear functions are such that an **inverse nonlinearity** can be found to be placed in series with it so the combination is linear. This method is often used to correct mild nonlinear characteristics of a sensor or actuator that have small variations in use.

The magnitude of the values of the variables in a problem is often very different, sometimes so much so that numerical difficulties arise. This was a serious problem years ago when equations were solved using analog computers,

[11] In 1949 the Russian scientist Aizerman conjectured that if a certain class of systems were stable with any linear gain between two limits, then the nonlinear system with a nonlinear gain characteristic that was kept between the same limits would also be stable. Unfortunately, this conjecture is not true.

and it was routine to *scale* the variables so that all had similar magnitudes. Today's widespread use of digital computers for solving differential equations has largely eliminated the need to scale a problem unless the number of variables is very large because computers are now capable of accurately handling numbers with wide variations in magnitude. Nevertheless, it is wise to understand the principle of scaling for the few cases where extreme variations in magnitude exist and scaling is necessary or the computer wordsize is limited. Sections 2.6.3 and 2.6.4 discuss two kinds of scaling.

### 2.6.1 Small-Signal Linearization

A nonlinear differential equation is one where the derivatives of the state have a nonlinear relationship to the state itself and/or the control. In other words, the differential equations *cannot* be written in the form

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u$$

but must be left in the form[12]

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u).$$

For small-signal linearization we first determine equilibrium values of $\mathbf{x}_o$, $\mathbf{u}_o$, that is, values where $\dot{\mathbf{x}}_o = \mathbf{0} = \mathbf{f}(\mathbf{x}_o, u_o)$. We then expand the nonlinear equation in terms of perturbations from these equilibrium values; that is, we let $\mathbf{x} = \mathbf{x}_o + \delta\mathbf{x}$ and $u = u_o + \delta u$, so that

$$\dot{\mathbf{x}}_o + \delta\dot{\mathbf{x}} \cong \mathbf{f}(\mathbf{x}_o, u_o) + \mathbf{F}\delta\mathbf{x} + \mathbf{G}\delta u,$$

where $\mathbf{F}$ and $\mathbf{G}$ are the best linear fits to the nonlinear function $\mathbf{f}(\mathbf{x}, u)$ at $\mathbf{x}_o$ and $u_o$, that is,

$$\mathbf{F} = \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{x_o, u_o} \quad \text{and} \quad \mathbf{G} = \left[\frac{\partial \mathbf{f}}{\partial u}\right]_{x_o, u_o}. \tag{2.101}$$

Substracting out the equilibrium solution, this reduces to

$$\delta\dot{\mathbf{x}} = \mathbf{F}\delta\mathbf{x} + \mathbf{G}\delta u, \tag{2.102}$$

which is a linear differential equation approximating the dynamics of the motion *about* the equilibrium point. Normally, the $\delta$ notation is dropped and it is understood that $x$ and $u$ refer to the deviation from the equilibrium.

In developing the models discussed so far in this chapter, we have encountered nonlinear equations on several occasions: the pendulum in Example 2.5, the hanging crane in Example 2.6, the AC induction motor in Section 2.4, the

[12] This equation assumes the system is time-invariant. A more general expression would be $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u, t)$.

tank flow in Example 2.20, and the hydraulic actuator in Example 2.21. In each case, we assumed either that the motion was small or that motion from some operating point was small, so that nonlinear functions were approximated by linear functions. The steps followed in those examples essentially involved finding **F** and **G** in order to linearize the differential equations to the form of Eq. (2.102) as illustrated in the following example.

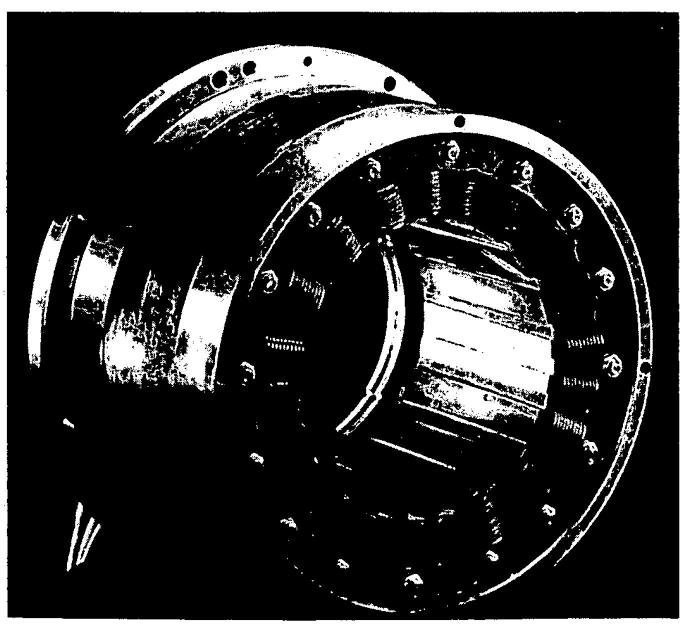
## EXAMPLE 2.22 *Linearization of Motion in a Ball Levitator*

Figure 2.34 shows a magnetic bearing used in large turbo machinery. The magnetics are energized using feedback control methods so that the axle is always in the center and never touches the magnets, thus keeping friction to an almost nonexistent level. A simplified version of a magnetic bearing that can be built in a laboratory is shown in Fig. 2.35, where one electromagnet is used to levitate a metal ball. The physical arrangement of the levitator is depicted in Fig. 2.36. The equation of motion of the ball, derived from Newton's law, Eq. (2.1), is

$$m\ddot{x} = f_m(x, i) - mg, \tag{2.103}$$

where the force $f_m(x, i)$ is caused by the field of the electromagnet. Theoretically, the force from an electromagnet falls off with an inverse square relationship to the distance from the magnet, but the exact relationship for the laboratory levitator is difficult to derive from physical principles because its magnetic field is so complex. However, the forces can be measured with a scale. Figure 2.37 shows the experimental curves for a ball with a 1-cm diameter and a mass of $8.4 \times 10^{-3}$ kg. At the value for the current of $i_2 = 600$ mA and the displacement $x_1$ shown in the figure, the magnetic force $f_m$ just cancels the gravity force $mg = 82 \times 10^{-3}$ N. (The mass of the ball is

**Figure 2.34**
A magnetic bearing *(Photo courtesy of Magnetic Bearings, Inc.)*



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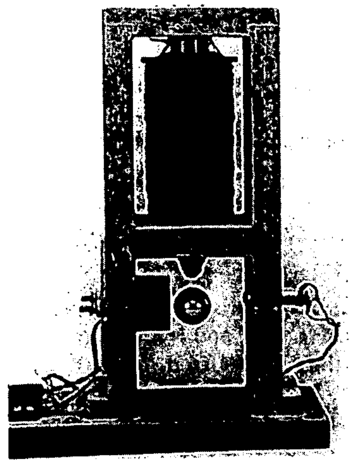

**Figure 2.35**
Magnetic ball levitator used in the laboratory

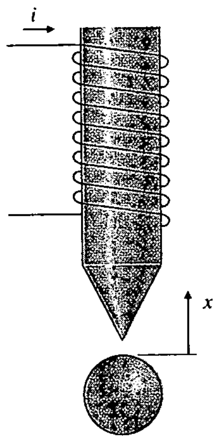


**Figure 2.36**
Model for ball levitation

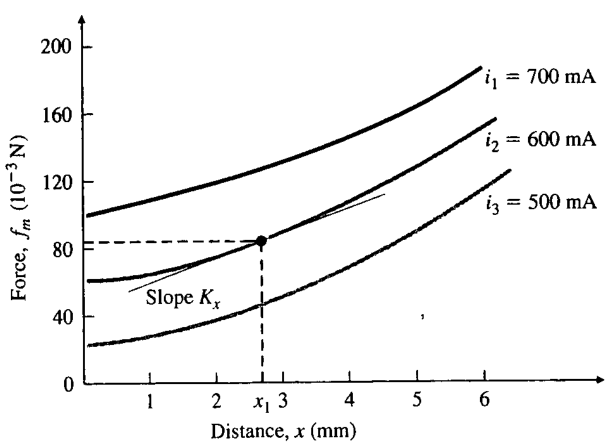


**Figure 2.37**
Experimentally determined force curves

EXAMPLE

$8.4 \times 10^{-3}$ kg, and the acceleration of gravity is 9.8 m/sec$^2$.) Therefore the point $(x_1, i_2)$ represents an equilibrium. Using the data, find the linearized equations of motion about the equilibrium point.

**Solution.** First we write in expansion form the force in terms of deviations from the equilibrium values $x_1$ and $i_2$:

$$f_m(x_1 + \delta x, i_2 + \delta i) \cong f_m(x_1, i_2) + K_x \delta x + K_i \delta i. \tag{2.104}$$

The linear gains are found as follows: $K_x$ is the slope of the force versus $x$ along the curve $i = i_2$, as shown in Fig. 2.37, and is found to be about 14 N/m. $K_i$ is the change of force with current for the value of fixed $x = x_1$. We find that for $i = i_1 = 700$ mA at $x = x_1$, the force is about $122 \times 10^{-3}$ N, and at $i = i_3 = 500$ mA at $x = x_1$, it is about $42 \times 10^{-3}$ N. Thus

$$\begin{aligned} K_i &\cong \frac{122 \times 10^{-3} - 42 \times 10^{-3}}{700 - 500} = \frac{80 \times 10^{-3}\text{ N}}{200\text{ mA}} \\ &\cong 400 \times 10^{-3}\text{ N/A} \\ &\cong 0.4\text{ N/A}. \end{aligned}$$

Substituting these values into Eq. (2.104) leads to the following linear approximation for the force in the neighborhood of equilibrium:

$$f_m \cong 82 \times 10^{-3} + 14\delta x + 0.4\delta i.$$

Substituting this expression into Eq. (2.103) and using the numerical values for mass and gravity force, we get for the linearized model

$$(8.4 \times 10^{-3})\ddot{x} = 82 \times 10^{-3} + 14\delta x + 0.4\delta i - 82 \times 10^{-3}.$$

Because $x = x_1 + \delta x$, then $\ddot{x} = \delta\ddot{x}$. The equation in terms of $\delta x$ is thus

$$\begin{aligned} (8.4 \times 10^{-3})\delta\ddot{x} &= 14\delta x + 0.4\delta i, \\ \delta\ddot{x} &= 1667\delta x + 47.6\delta i, \end{aligned} \tag{2.105}$$

which is the desired linearized equation of motion about the equilibrium point. A logical state vector is $\mathbf{x} = [\delta x\ \delta\dot{x}]^T$, which leads to the standard matrices

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ 1667 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{G} = \begin{bmatrix} 0 \\ 47.6 \end{bmatrix}$$

and the control $u = \delta i$.

EXAMPLE 2.23

### *Alternate Linearization of the Water Tank*

Repeat the linearization of Example 2.20 using the concepts presented in this section.

**Solution.** Equation (2.87) may be written as

$$\dot{x} = f(x, u) \tag{2.106}$$

where $x \triangleq h$, $u \triangleq w_{in}$, and $f = -\frac{1}{RA\rho}\sqrt{p_1 - p_a} + \frac{1}{A\rho}w_{in} = -\frac{1}{RA\rho}\sqrt{\rho g h - p_a} + \frac{1}{A\rho}w_{in}$. The linearized equations are of the form

$$\delta\dot{x} = F\delta x + G\delta u, \tag{2.107}$$

where

$$[F]_{x_o,u_o} = \frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial h}\right]_{h_o,u_o} = \frac{\partial}{\partial h}\left[-\frac{1}{RA\rho}\sqrt{\rho g h - p_a}\right]_{h_o,u_o} \tag{2.108}$$

$$= -\frac{g}{2AR}\frac{1}{\sqrt{\rho g h_o - p_a}} = -\frac{g}{2AR}\frac{1}{\sqrt{p_o - p_a}} \tag{2.109}$$

and

$$[G]_{x_o,u_o} = \frac{\partial f}{\partial u} = \frac{\partial f}{\partial w_{in}} = \frac{1}{A\rho}. \tag{2.110}$$

However, note that some flow is required to maintain the system in equilibrium so that Eq. (2.107) is valid; specifically, we see from Eq. (2.87) that

$$u_o = w_{in_o} = \frac{1}{R}\sqrt{p_o - p_a} \qquad \text{for } \dot{h} = 0, \tag{2.111}$$

and the $\delta u$ in Eq. (2.107) is $\delta w_{in}$, where $w_{in} = w_{in_o} + \delta w_{in}$. Therefore, Eq. (2.107) becomes

$$\delta\dot{h} = F\delta h + G\delta w_{in} = F\delta h + G w_{in} - G\frac{1}{R}\sqrt{p_o - p_a} \tag{2.112}$$

and matches Eq. (2.90) precisely.

---

## 2.6.2 Linearization by Feedback

Linearization by feedback is accomplished by substracting the nonlinear terms out of the equations of motion and adding them to the control. The result is a linear system, provided that the computer implementing the control has enough capability to compute the nonlinear terms fast enough. A more detailed understanding of the method is best achieved through example.

To illustrate linearization by feedback, we consider the equation of a simple pendulum developed in Example 2.5 [Eq. (2.21)]:

$$ml^2\ddot{\theta} + mgl\sin\theta = T_c. \tag{2.113}$$

If we compute the torque $T_c$ to be

$$T_c = mgl\sin\theta + u, \tag{2.114}$$

then the motion is described by

$$ml^2\ddot{\theta} = u. \tag{2.115}$$

Equation (2.115) is a linear equation *no matter how large the angle $\theta$ becomes*. We use it as the model for purposes of control design because it enables us to use linear analysis techniques. The resulting linear control will provide the value of $u$ based on measurements of $\theta$; however, the value of the torque actually sent to the equipment would derive from Eq. (2.114). For robots with two or three rigid links, this computed-torque approach has led to effective control. It is also being researched for the control of aircraft, where the linear models change considerably in character with the flight regime.

### 2.6.3 Amplitude Scaling

There are two types of scaling that are sometimes carried out: amplitude scaling and time scaling. **Amplitude scaling** is usually performed unwittingly by simply picking units that make sense for the problem at hand. For the ball levitator, expressing the motion in millimeters and the current in milliamps would keep the numbers within a range that is easy to work with. Equation (2.105) was developed in the standard SI units of meters, kilograms, and amperes, but in computing the motion of a rocket going into orbit, using kilometers makes more sense. The equations of motion are usually solved using computer-aided design software, which is often capable of working in any units. For higher-order systems it becomes important to scale the problem so that the elements of the state vector have similar numerical variations. A method for accomplishing the best scaling for a complex system is first to estimate the maximum values for each state element and then to scale the system so that each element varies between $-1$ and 1.

In general, we can perform amplitude scaling by defining the scaled variables for each state element: If

$$x' = S_x x, \tag{2.116}$$

then

$$\dot{x}' = S_x\dot{x} \quad \text{and} \quad \ddot{x}' = S_x\ddot{x}. \tag{2.117}$$

We then pick $S_x$ to result in the appropriate scale change, substitute Eqs. (2.116) and (2.117) into the equations of motion, and recompute the coefficients.

EXAMPLE 2.24 ***Scaling for the Ball Levitator***

Scale the variables for the ball levitator in Example 2.22 to result in units of millimeters and milliamps instead of meters and amps.

**Solution.** Referring to Eq. (2.116), we define

$$\delta x' = S_x \delta x \quad \text{and} \quad \delta i' = S_i \delta i$$

such that both $S_x$ and $S_i$ have a value of 1000 in order to convert $\delta x$ and $\delta i$ in meters and amps to $\delta x'$ and $\delta i'$ in millimeters and milliamps. Substituting these relations into Eq. (2.105) and taking note of Eq. (2.117) yields

$$\delta \ddot{x}' = 1667 \delta x' + 47.6 \frac{S_x}{S_i} \delta i'.$$

In this case $S_x = S_i$, so Eq. (2.105) remains unchanged. Had we scaled the two quantities by different amounts, there would have been a change in the last coefficient in the equation.

### 2.6.4 Time Scaling

The unit of time when using SI units or English units is seconds. Computer-aided design software is *usually* able to compute results accurately no matter how fast or slow the particular problem at hand. However, if a dynamic system responds in a few microseconds or if there are characteristic frequencies in the system on the order of several MHz, the problem may become ill-conditioned, so that the numerical routines produce errors. This can be particularly troublesome for high-order systems. The same holds true for an extremely slow system. It is therefore useful to know how to change the units of time should you encounter an ill-conditioned problem.

We define the new scaled time to be

$$\tau = \omega_o t \tag{2.118}$$

such that, if $t$ is measured in seconds and $\omega_o = 1000$, then $\tau$ will be measured in milliseconds. The effect of the time scaling is to change the differentiation so that

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d(\tau/\omega_o)} = \omega_o \frac{dx}{d\tau}, \tag{2.119}$$

and

$$\ddot{x} = \frac{d^2x}{dt^2} = \omega_o^2 \frac{d^2x}{d\tau^2}. \tag{2.120}$$

Putting the equation into state-variable form allows a more concise way of stating time scaling. For the system described by

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u, \tag{2.121}$$

we say it is time-scaled, using $\tau = \omega_o t$, by the equations

$$\frac{d\mathbf{x}}{d\tau} = \frac{1}{\omega_o}\mathbf{F}\mathbf{x} + \frac{1}{\omega_o}\mathbf{G}u. \tag{2.122}$$

### EXAMPLE 2.25 *Time Scaling an Oscillator*

The equation for an oscillator was derived in Example 2.5. For a case with a very fast natural frequency $\omega_n = 15{,}000$ rad/sec (about 2 kHz), Eq. (2.23) can be rewritten as

$$\ddot{\theta} + 15{,}000^2 \cdot \theta = 10^6 \cdot T_c.$$

Determine the time-scaled equation so that the unit of time is milliseconds.

**Solution.** The value of $\omega_o$ in Eq. (2.118) is 1000. Equation (2.80) shows that

$$\frac{d^2\theta}{d\tau^2} = 10^{-6} \cdot \ddot{\theta},$$

and the time-scaled equation becomes

$$\frac{d^2\theta}{d\tau^2} + 15^2 \cdot \theta = T_c$$

In practice, we would then solve the equation

$$\ddot{\theta} + 15^2 \cdot \theta = T_c \tag{2.123}$$

and label the plots in milliseconds instead of seconds.

In state-variable form with a state vector $\mathbf{x} = [\theta\ \dot{\theta}]^T$, the unscaled matrices are

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ -15{,}000^2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{G} = \begin{bmatrix} 0 \\ 10^6 \end{bmatrix}.$$

Applying Eq. (2.122) results in

$$\mathbf{F}' = \begin{bmatrix} 0 & \frac{1}{1000} \\ -\frac{15{,}000^2}{1000} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{G}' = \begin{bmatrix} 0 \\ 10^3 \end{bmatrix},$$

which yields state-variable equations which are equivalent to the scaled system in Eq. (2.123).

## SUMMARY

- Mathematical modeling of the system to be controlled is the first step in analyzing and designing the required system controls. In this chapter we developed models for representative systems. Important equations for each category of system are summarized in Table 2.1.
- An alternative way of expressing the differential equations that characterize the model of a linear system is the **state-variable form**,

$$\dot{\mathbf{x}} = \mathbf{Fx} + \mathbf{G}u,$$

$$y = \mathbf{Hx} + Ju.$$

TABLE 2.1 **Key Equations for Dynamic Models**

| System | Important Laws or Relationships | Associated Equations | Equation Number |
|---|---|---|---|
| Mechanical | Translation motion (Newton's law) | $\mathbf{F} = ma$ | (2.1) |
| | Rotational motion | $M = I\alpha$ | (2.14) |
| | Motion of nonrigid bodies | $m\ddot{x} + b\dot{x} + kx = F$ | (2.10) |
| State-variable form | Linear system | $\dot{\mathbf{x}} = \mathbf{Fx} + \mathbf{G}u$ | (2.39) |
| | | $y = \mathbf{Hx} + Ju$ | (2.40) |
| Electrical | Operational amplifier | | (2.47), (2.48) |
| Electromechanical | Law of motors | $F = Bli$ | (2.54) |
| | Law of the generator | $e(t) = Blv$ | (2.57) |
| | Torque developed in a rotor | $T = K_t i_a$ | (2.61) |
| Back emf | Voltage generated as a result of rotation of a rotor | $e = K_e \dot{\theta}_m$ | (2.62) |
| Heat flow | Heat-energy flow | $q = \frac{1}{R}(T_1 - T_2)$ | (2.73) |
| | Temperature as a function of heat-energy flow | $\dot{T} = \frac{1}{C}q$ | (2.74) |
| | Specific heat | $C = mc_v$ | (2.75) |
| Fluid flow | Continuity relation (conservation of matter) | $\dot{m} = w_{\text{in}} - w_{\text{out}}$ | (2.81) |
| | Force of a fluid acting on a piston | $f = pA$ | (2.83) |
| | Effect of resistance to fluid flow | $w = \frac{1}{R}(p_1 - p_2)^{1/\alpha}$ | (2.84) |

Equations in state-variable form are conducive to solution by computer packages that were developed especially for matrix equations (e.g., MATLAB). In Section 2.2 we briefly introduced the state-variable form; it will be explored in more depth in Chapter 7.

- **Linearization** and **scaling** (Section 2.6) are methods by which certain complications of dealing with differential equations can be minimized. In linearization, nonlinear differential equations are approximated by linear ones by either (1) considering a small-signal linear model that is acurate near an equilibrium, or (2) linearization by feedback, or (3) introducing an inverse nonlinearity. Scaling of variables results in numerical values that fall within a narrow-enough range of magnitude to minimize errors and allow for ease of computation.

## Review Questions

1. What is a "free-body" diagram?
2. What are the two forms of Newton's law?
3. Why is it convenient to write equations of motion in the state-variable form?
4. For a structural process to be controlled such as a robot arm, what is the meaning of "collocated control"? "Noncollocated control"?
5. When, why, and by whom was the device named an "operational amplifier"?
6. What is the major benefit of having zero input current to an operational amplifier?
7. State Kirchoff's Current Law.
8. State Kirchoff's Voltage Law.
9. Why is it important to have a small value for the armature resistance, $R_a$, of an electric motor?
10. What are the definition and units of the electric constant of a motor?
11. What are the definition and units of the torque constant of an electric motor?
12. Give the relationships for (a) heat flow across a substance and (b) heat storage in a substance.
13. Name and give the equations for the three relationships governing fluid flow.
14. Why do we approximate a physical model of the plant (which is *always* nonlinear) with a linear model?

## Problems

### Problems for Section 2.1

**2.1.** Write the differential equations for the mechanical systems shown in Fig. 2.38.

**2.2.** Write the equations of motion of a pendulum consisting of a thin, 2-kg stick of length $l$ suspended from a pivot. How long should the rod be in order for the period to be exactly 2 sec? (The inertia $I$ of a thin stick about an endpoint is $\frac{1}{3}ml^2$. Assume $\theta$ is small enough that $\sin\theta \cong \theta$.)